Smooth manifold

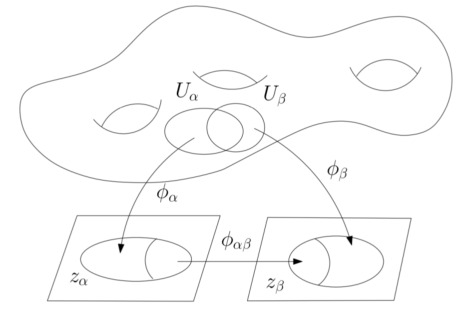
Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.



Definition (Tangent Vector)

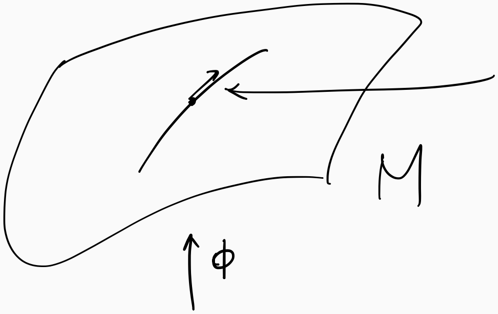
A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

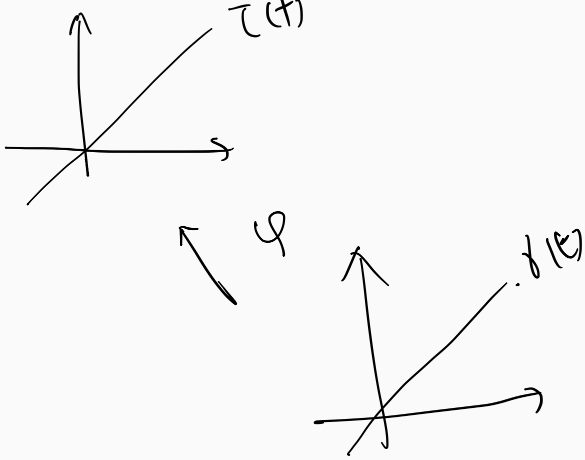
$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

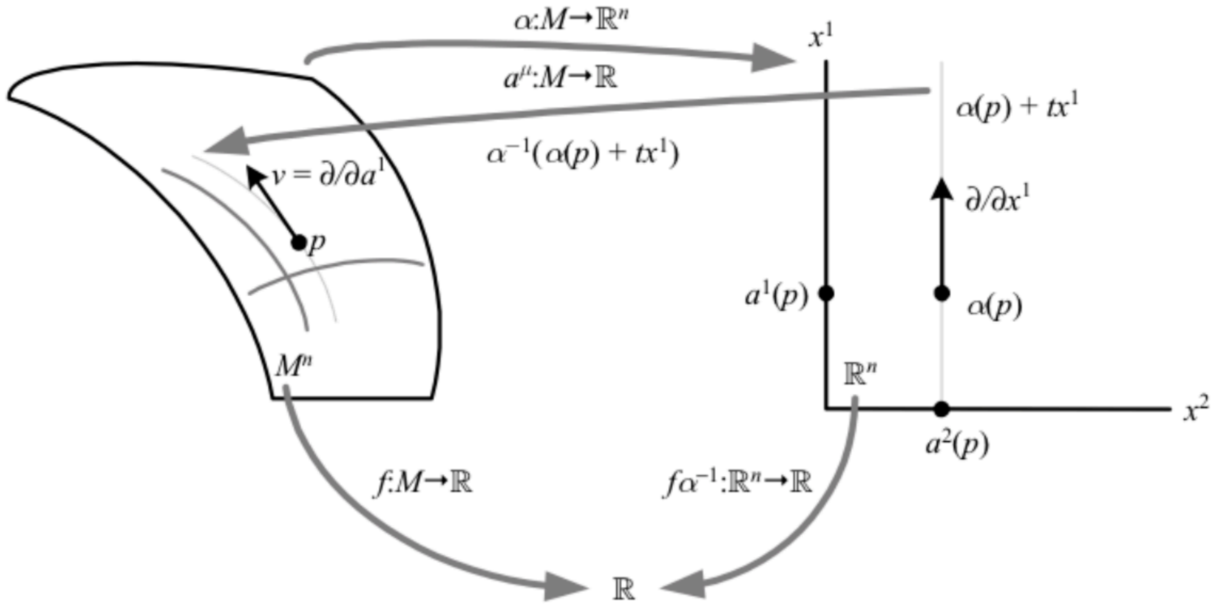
$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.



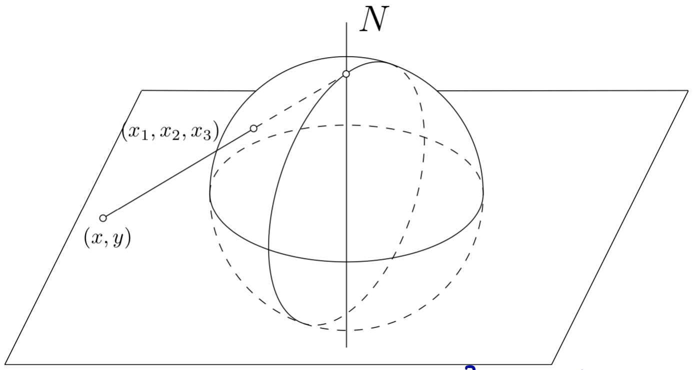
$$\frac{d}{dt}\Big|_{t=0} \phi(\tau(t))$$

$$= \frac{d}{dt}\Big|_{t=0} \phi(\tau \circ \gamma(t))$$





$\alpha: M \to \mathbb{R}^n$
$a^\mu: M \to \mathbb{R}$
$\alpha^{-1}(\alpha(p) + tx^1)$
$v = \partial/\partial a^1$
p
M^n
x^1
$\alpha(p) + tx^1$
$\partial/\partial x^1$
$a^1(p)$
$\alpha(p)$
$\mathbb{R}^n$
x^2
$a^2(p)$
$f: M \to \mathbb{R}$
$f\alpha^{-1}: \mathbb{R}^n \to \mathbb{R}$
$\mathbb{R}$



Sterographic projection from $\mathbb{C}$ to $\mathbb{S}^2 \setminus \{N\}$.

$\phi : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$ defined by:

$$\phi(x, y) = (x_1, x_2, x_3) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right)$$

$$\phi^{-1}(x_1, x_2, x_3) = (x, y) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$$

$$\frac{\partial}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{2}{(1+x^2+y^2)^2}(1-x^2+y^2, \ -2xy, \ 2x)$$

$$\frac{\partial}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{2}{(1+x^2+y^2)^2}(-2xy, \ 1+x^2-y^2, \ 2y)$$

Note that:

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = \frac{4}{(1+x^2+y^2)^2}$$

$$\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = \frac{4}{(1+x^2+y^2)^2}$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = 0$$

Angle preserving!

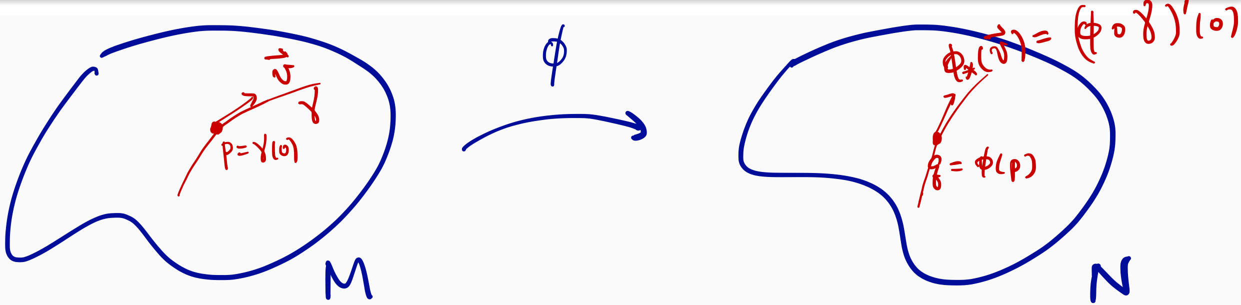
$$\frac{d}{dt}\phi(x+t, y)$$

Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from M to N, $\gamma : (-\epsilon, \epsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_pM$, then $\phi \circ \gamma$ is a curve on N, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)}N,$$

as the push-forward tangent vector of $\mathbf{v}$ induced by ϕ.



Integration on surface

Definition: Suppose $\mathcal{U} \subset M$ is an open set of a 2-dim manifold M, and $\phi : \mathcal{U} \to \Omega \subset \mathbb{R}^2$ is a chart. Then:

$$\int_{\mathcal{U}} f \, dA = \int_{\Omega} f \circ \phi^{-1} \sqrt{EG - F^2} \, du \, dv$$

where $E = (\phi^{-1})_u \cdot (\phi^{-1})_u$, $F = (\phi^{-1})_u \cdot (\phi^{-1})_v$, $G = (\phi^{-1})_v \cdot (\phi^{-1})_v$

Definition: Choose a partition of unity $\{\psi_i : \mathcal{U}_i \to \mathbb{R}\}_{i \in I}$ such that $\bigcup_i \mathcal{U}_i = M$, $\psi_i(p) \geq 0$ for $\forall i$ and $\sum_i \psi_i(p) \equiv 1$ for $\forall p \in M$.

Then:
$$\int_M f \, dA = \sum_i \int_{\mathcal{U}_i} \psi_i f \, dA$$
$$= \sum_i \int_{\Omega_i} (\psi_i f) \circ \phi_i^{-1} \sqrt{EG - F^2} \, du \, dv$$

where $\phi_i : \mathcal{U}_i \to \Omega_i$ is a chart.

Gauss-Bonnet Theorem

$\frac{d}{dt}\Big|_{t=0} \left(\vec{N}(\phi(\gamma(t))\right)$, $\phi(\gamma(0)) = p$, $\frac{d}{dt}\Big|_{t=0} \phi\circ\gamma(t) = \vec{v}$

Definition: Let $p \in M$ and $\vec{v} \in T_pM$ (tangent plane at p).

Define: $S_p(\vec{v}) = -D_{\vec{v}}\vec{N}$, where $\vec{N}$ is the normal direction of M at p. Then: $S_p : T_pM \to T_pM$ is a linear operator, called the shape operator.

The Gaussian curvature at p is defined as:

$$K = \det(S_p).$$

Theorem: (Gauss-Bonnet) Let M be a compact closed surface.

$$\int_M K\, dA = 2\pi \underbrace{\chi(M)}_{\text{Euler characteristic}}$$

Euler characteristic
(integer depending on the topology)

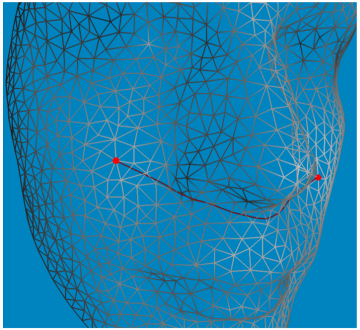
Discrete Gauss-Bonnet Theorem

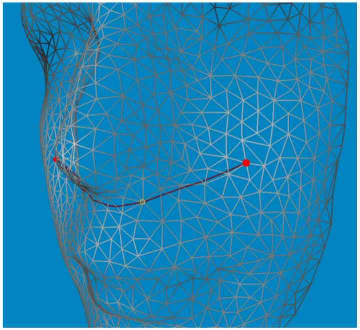
Theorem: For an oriented discrete triangulated surface M,

$$\sum_{v_i} K(v_i) = 2\pi\,\chi(M)$$

where $\{v_i\}$ is the collection of vertices, $K(\vec{v}_i)$ is the discrete Gaussian curvature defined as: $K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases}$

and $\chi(M) = \underbrace{|V|}_{\text{\# of vertices}} + \overbrace{|F|}^{\text{\# of faces}} - \underbrace{|E|}_{\text{\# of edges}}$





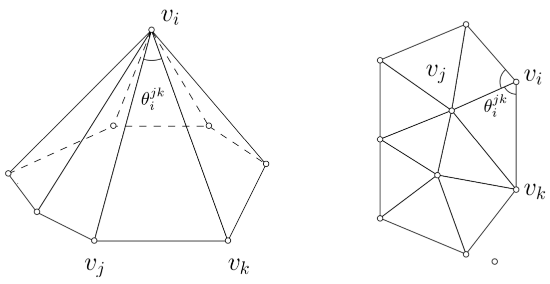


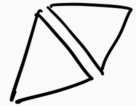
Figure: Discrete Gaussian curvature.

<u>Proof:</u> Let $M = (V, E, F)$. If M is closed, then:

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jk}$$

$$= 2\pi |V| - \pi |F|$$

$\because$ M is closed $\therefore$ $3|F| = 2|E|$



$$\therefore \chi(M) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F|$$

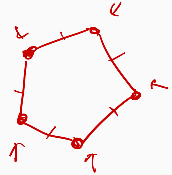
$$= |V| - \frac{1}{2}|F|$$

$$\therefore \sum_{v_i \in V} K(v_i) = 2\pi \chi(M).$$

Assume M has a boundary ∂M.

Let V_0 = interior vertex set
V_1 = boundary set
} $|V| = |V_0| + |V_1|$

E_0 = interior edge set
E_1 = boundary edge set
} $|E| = |E_0| + |E_1|$



$\because$ All boundary are closed loop $\therefore$ $|E_1| = |V_1|$.

Each interior edge is adjacent to two faces and each boundary edge is adjacent to one face, we have:

$$3|F| = 2|E_0| + |E_1| = 2|E_0| + |V_1|$$

〃 $|V_1|$

$$\therefore \chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1|$$

$$= |V_0| + |F| - |E_0|$$

$\because$ $|E_0| = \frac{1}{2}(3|F| - |V_1|)$

$$\therefore \chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|.$$

$$\therefore \sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) = \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk} \right)$$

$$= 2\pi |V_0| + \pi |V_1| - \pi |F|$$

$$= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right)$$

$$= 2\pi \chi(M)$$

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Basic theories of compact Riemann surface

Definition: (Harmonic function) Suppose $u: D \to \mathbb{R}$ is a real valued function defined on $D \subseteq \mathbb{C}$. If $u \in C^2(D)$ and for any $z \in D$, $z = x + iy$, we have:

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \quad \text{for } \forall z.$$

Then: u is a harmonic function.

Definition: (Holomorphic function) A function $f: \mathbb{C} \to \mathbb{C}$, $(x, y) \mapsto (u, v)$ is holomorphic if:

$$\begin{cases} \dfrac{\partial u}{\partial x}(z) = \dfrac{\partial v}{\partial y}(z) \\[2ex] \dfrac{\partial u}{\partial y}(z) = -\dfrac{\partial v}{\partial x}(z) \end{cases} \quad \text{for } \forall z \in \mathbb{C}$$

(Cauchy-Riemann eqt)

Remark: • Denote $dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Then: $\Delta = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ (Check!)

Also, f is holomorphic if $\dfrac{\partial f}{\partial \bar{z}} = 0$. (Check!)

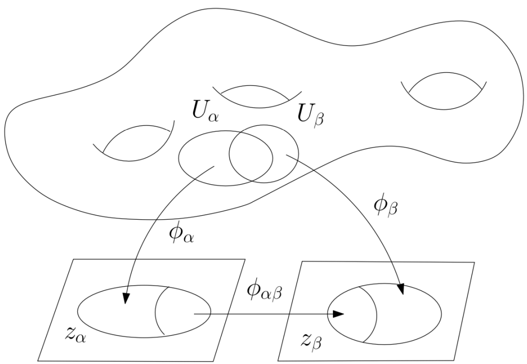
• If a holomorphic function is bijective and f^{-1} is also holomorphic, then f is called biholomorphic or conformal.

Definition: (Riemann surface) A Riemann surface S is a 2-dim manifold M with an atlas $\{(U_\alpha, z_\alpha)\}$, such that $\{U_\alpha\}$ is an open covering, $M \subset \bigcup_\alpha U_\alpha$ and $z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism from U_α to an open set in $\mathbb{C}$, $z_\alpha(U_\alpha)$. Also, if $U_\alpha \cap U_\beta \neq \phi$, then:

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$$

is biholomorphic / conformal.

$\{(U_\alpha, z_\alpha)\}$ is called the conformal atlas of S.



Remark: • Given two conformal atlas $\{(U_\alpha, z_\alpha)\}$ and $\{(V_\beta, \tau_\beta)\}$, if their union is also a conformal atlas, then we say $\{(U_\alpha, z_\alpha)\}$ is equivalent to $\{(V_\beta, \tau_\beta)\}$.

Each equivalence class of conformal atlas is called a <u>conformal structure.</u>

• Given a smooth manifold M, we can equip M with a Riemannian metric $g = (g_{ij})$, which gives the inner product in the tangent space $T_p(M)$,

$$g_{ij} = \langle \partial_i, \partial_j \rangle_g .$$

Its inverse matrix is (g^{ij}), satisfies

$$\sum_{j=1}^{n} g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

• Suppose M has a Riemannian metric g. Then we require that on each chart of $\{(U_\alpha, z_\alpha)\}$:

$$g = e^{2\lambda(z_\alpha)} dz_\alpha d\overline{z_\alpha} = e^{2\lambda(z_\alpha)}(dx_\alpha^2 + dy_\alpha^2)$$

Recall: given $\vec{v} = v_1 \frac{\partial}{\partial x_\alpha} + v_2 \frac{\partial}{\partial y_\alpha} \in T_pM$

$\vec{w} = w_1 \frac{\partial}{\partial x_\alpha} + w_2 \frac{\partial}{\partial y_\alpha} \in T_pM$

Then: $(dx_\alpha^2 + dy_\alpha^2)(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2$

In this case, we say the local parameters associated to $\{(U_\alpha, z_\alpha)\}$ are isothermal coordinates.